Journal of Engineering Physics and Thermophysics, Vol. 71, No. 5, 1998

CALCULATION OF SYMMETRIC AND NONSYMMETRIC DRAINAGE OF A LIQUID FROM A VOLUME ON THE BASIS OF THE NAVIER–STOKES EQUATIONS

A. A. Prikhod'ko

UDC 532.516

For numerical simulation of drainage of a liquid from a volume of arbitrary cross section the Navier–Stokes equations for an incompressible liquid are used. Formulation of and an algorithm for solution of the initial-boundary-value problem and results of calculations of drainage from symmetric and nonsymmetric volumes are considered. Isolines of the Cartesian velocity components, distributions of the velocity vectors, and the position of the free surface for different moments of time are presented. Calculations are performed within the framework of a package of applied programs developed by the author for solving the of Euler and Navier–Stokes equations.

1. Introduction. Success in the field of hydromechanics has been achieved recently due to improvements in numerical methods of solving the Navier–Stokes equations and due to an increase in the memory and speed of computers. A wide circle of problems relates to investigation of the motion of a liquid with free surfaces [1-5]. In most works touching upon drainage of liquids from volumes the model of vortex-free flows of an incompressible liquid [4] or the Stokes model for slow flows of high-viscosity liquids [5] was used. Results of numerical simulation of the drainage of a viscous incompressible liquid from volumes of different cross sections on the basis of the Navier–Stokes equations with the change in the shape of the free surface taken in consideration are given below. A versatile package of applied programs developed by the author for numerical solution of the Navier–Stokes and Euler equations [6] was used in the calculation.

2. Formulation of the Problem. By introducing scales for the quantities sought and for the independent variables, the system of equations of motion of a viscous incompressible liquid can be brought into dimensionless form [3]:

$$\frac{\partial \overline{V}}{\partial t} + (\overline{V}\nabla)\overline{V} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta \overline{V} + \overline{n}g, \qquad \mathrm{div}\,\overline{V} = 0. \tag{1}$$

If the components of the velocity vector relative to a Cartesian system of coordinates are left as dependent variables and approximation of the Navier–Stokes equations for an incompressible liquid by means of the method of artificial compressibility [7] is used, then after passage to an arbitrary system of curvilinear coordinates the initial equations (1) for two-dimensional plane and axisymmetric flows can be written in the following manner:

$$\frac{\partial \hat{q}}{\partial t} + \frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} = \frac{1}{\mathrm{Re}}\left(\frac{\partial \hat{T}}{\partial \xi} + \frac{\partial \hat{S}}{\partial \eta}\right) + \hat{H}, \tag{2}$$

where

$$\hat{q} = \frac{1}{J}q, \quad \hat{E}, \hat{F} = \frac{1}{J}(k_0 q + k_1 E + k_2 F), \quad \hat{H} = \frac{1}{J}H,$$

Dnepropetrovsk State University, Ukraine. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 71, No. 5, pp. 855-859, September-October, 1998. Original article submitted May 27, 1997.

1062-0125/98/7105-0842$20.00

$$q = \begin{bmatrix} p \\ u \\ v \end{bmatrix}, \quad E = \begin{bmatrix} \beta u \\ u^2 + p \\ uv \end{bmatrix}, \quad F = \begin{bmatrix} \beta v \\ uv \\ v^2 + p \end{bmatrix},$$

$$\hat{T} = \frac{1}{J} \begin{bmatrix} 0 \\ \alpha u_\xi \\ \alpha v_\xi \end{bmatrix}, \quad \hat{S} = \frac{1}{J} \begin{bmatrix} 0 \\ \gamma u_\eta \\ \gamma v_\eta \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix},$$

$$\alpha = \xi_x^2 + \xi_y^2, \quad \gamma = \eta_x^2 + \eta_y^2,$$

$$\xi = \xi\,(x, y, t), \quad \eta = \eta\,(x, y, t), \quad J = \frac{D\,(\xi, \eta)}{D\,(x, y)} = \xi_x \eta_y - \eta_x \xi_y$$

is the Jacobian of the transformation of coordinates. The metric coefficients

$$\{k_0, k_1, k_2\} = \{\xi_t, \xi_x, \xi_y\} = \{\eta_t, \eta_x, \eta_y\}$$

are defined in terms of derivatives of the Cartesian coordinates of the points, which depend on the time, with the aid of the relations

$$\begin{aligned} \xi_x &= J y_\eta, \quad \xi_y = -J x_\eta, \quad \xi_t = -x_t \xi_x - y_t \xi_y, \\ \eta_x &= -J y_\xi, \quad \eta_y = J x_\xi, \quad \eta_t = -x_t \eta_x - y_t \eta_y. \end{aligned} \tag{3}$$

In calculating axisymmetric flows on the basis of the system of equations (2), the terms distinguishing the system of equations with axial symmetry from the plane one were included in the source term H. The metric coefficients (3) and the Jacobian of the coordinate transformation were modified accordingly.

The system of equations was closed by initial and boundary conditions. At the initial moment of time the liquid was at rest, and the volume was full. The shape of the free surface was flat. The flow in the drainage orifice was a given constant, with a profile in the form of the Poiseuille solution, or else the flow and the velocity profile were calculated in the process of solution. At the axis of the volume, conditions of axial symmetry must be fulfilled. Conditions of adhesion were given for the walls.

The position of the free surface was determined in the process of solution from the kinematic condition [2]

$$\frac{\partial f}{\partial t} + \bar{V} \nabla f = 0, \tag{4}$$

where $y = f(x, t)$ is the equation of the free surface. Here, dynamic boundary conditions [2-5] must be fulfilled:

$$\bar{s}\,\bar{\bar{\Pi}}\,\bar{n} = 0, \quad \bar{n}\,\bar{\bar{\Pi}}\,\bar{n} = -p_0. \tag{5}$$

Capillary forces were considered small in comparison to viscous and gravitational ones and were ignored.

In the calculation a system of arbitrary nonorthogonal coordinates was used. Mapping of the physical region (x, y) bounded by the walls of the volume, the axis of symmetry, the free surface, and the drainage orifice onto the square calculational region (ξ, η) is given by the equations of the transformation of coordinates $\xi = \xi(x, y, t)$, $\eta = \eta(x, y, t)$, where ξ and η are the solution of the Dirichlet boundary-value problem for the system of equations of elliptic type [8]

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = P\,(\xi, \eta), \quad \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = Q\,(\xi, \eta). \tag{6}$$

Here $P(\xi, \eta)$ and $Q(\xi, \eta)$ are the functions with whose aid regulation of the coordinate lines was implemented.

If the roles of the dependent (ξ, η) and independent (x, y) variables are exchanged, we arrive at a coupled system of equations:

$$A\frac{\partial^2 x}{\partial\xi^2} - 2B\frac{\partial^2 x}{\partial\xi\partial\eta} + C\frac{\partial^2 x}{\partial\xi^2} + J^{-2}\left[P(\xi,\eta)\frac{\partial x}{\partial\xi} + Q(\xi,\eta)\frac{\partial x}{\partial\xi}\right] = 0,$$

$$A\frac{\partial^2 y}{\partial\xi^2} - 2B\frac{\partial^2 y}{\partial\xi\partial\eta} + C\frac{\partial^2 y}{\partial\xi^2} + J^{-2}\left[P(\xi,\eta)\frac{\partial y}{\partial\xi} + Q(\xi,\eta)\frac{\partial y}{\partial\xi}\right] = 0, \tag{7}$$

in which $A = x_\eta^2 + y_\eta^2$, $B = x_\xi x_\eta + y_\xi y_\eta$, $C = x_\xi^2 + y_\xi^2$.

3. **Numerical Method.** Solution of the system of equations (7) for determination of the coordinates of the nodal points at each time step was implemented by the method of upper relaxation.

In solving the system of equations (2), an implicit factorized scheme of Beam–Warming type was used [9]:

$$\left(I + h\delta_\xi \hat{A}^n - h\,\mathrm{Re}^{-1}\delta_\xi \hat{M}^n\right) \times \left(I + h\delta_\eta \hat{B}^n - h\,\mathrm{Re}^{-1}\delta_\eta \hat{N}^n\right) \times \left(\hat{q}^{n+1} - \hat{q}^n\right) =$$

$$= -\Delta t\left[\delta_\xi \hat{E}^n + \delta_\eta \hat{F}^n - \mathrm{Re}^{-1}\left(\delta_\xi \hat{T}^n + \delta_\eta \hat{S}^n\right) - H\hat{d}^n\right], \tag{8}$$

where $\hat{A}$, $\hat{B}$, $\hat{M}$, $\hat{N}$ are Jacobi matrices obtained in linearizing the vectors $\hat{E}$, $\hat{F}$, $\hat{T}$, $\hat{S}$ relative to the values at the previous time step:

$$\hat{A} = \hat{B} = \begin{bmatrix} k_0 & \beta k_1 & \beta k_2 \\ k_1 & \theta + k_1 u & u k_2 \\ k_2 & v k_1 & \theta + k_2 v \end{bmatrix}, \quad \theta = k_0 + k_1 u + k_2 v,$$

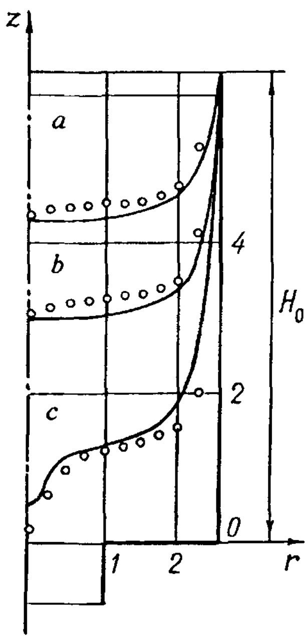
$$\hat{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha\frac{\partial}{\partial\xi} & 0 \\ 0 & 0 & \alpha\frac{\partial}{\partial\xi} \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma\frac{\partial}{\partial\eta} & 0 \\ 0 & 0 & \gamma\frac{\partial}{\partial\eta} \end{bmatrix}.$$

In Eqs. (8) h can be equal to Δt or $\Delta t/2$. For solving the matrix system of linear algebraic equations the method of vector run was used.

The suggested approach to simulation of the dynamics of a liquid with moving boundaries reduces to making use of one of the numerical methods of solving the Navier–Stokes equations in arbitrary curvilinear coordinates and an algorithm for singling out the moving boundaries. This approach can be implemented with the aid of a package of applied programs [6] developed by the author. Testing of the numerical methods used was carried out on the problem of the interaction of a compression jump with a laminar boundary layer [10, 11]. Here, numerical algorithms were selected so as to compare the implicit, explicit, and mixed implicit-explicit difference schemes lying at the basis of the method. Results of the calculations, expenditure of machine time, and the Courant number were compared. The algorithm for singling out the moving boundaries was tested on the problem of singling out shock waves in a supersonic flow around a circular cone at a large angle of attack [12, 13].

4. **Discussion of the Results.** In the present work, results of calculations obtained with the aid of the Beam–Warming difference scheme [9] are represented. Repetition of the calculations with the aid of other difference schemes available from the package of applied programs (predictor-corrector, explicit-implicit, increased accuracy, etc.) did not present great difficulties.

To check the workability of the approach considered above for calculation of drainage of a liquid from a volume on the basis of the Navier–Stokes equations, test calculations of drainage from a cylindrical volume investigated theoretically and experimentally in [4] were performed. Profiles of the free surface for three moments of time are given below. Experimental results [4] (points) and calculations of the present work (curves) are given in Fig. 1. In the calculations the dimensionless combination $W = \rho g R^2/\mu U$, which characterizes the ratio between the gravitational and viscous forces, was $W = 4.85$, and the dimensionless height of filling $H_0 = H/R = 6.25$; results are given for different moments of time with a narrowing coefficient, which characterizes the ratio of the cylindrical volume's radius to the drainage-pipe radius, $H_0/r = 2.5$.



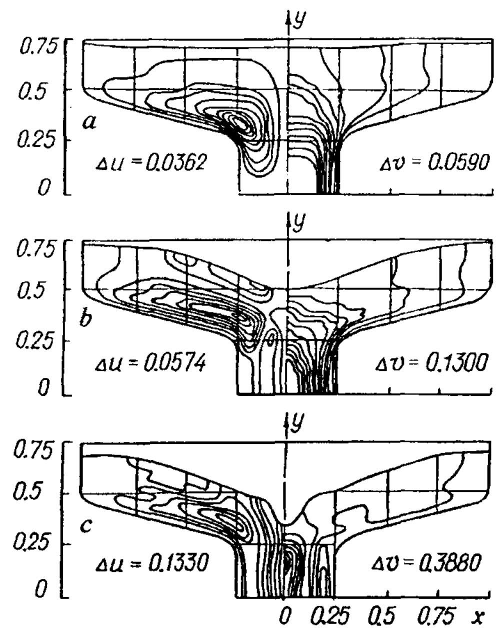
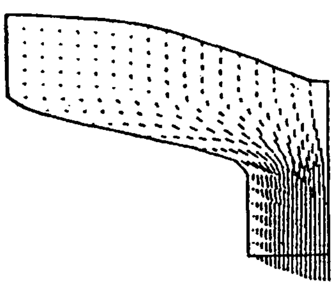


Fig. 1. Position of the free surface during drainage from a volume with axial symmetry (experiment from [4]): a) $t = 8.9$, b) 15.6, c) 26.0.

Fig. 2. Shape of the free surface and distribution of the Cartesian components of the velocity vector during drainage from a contoured volume: a) $t = 0.5$, b) 1.0, c) 1.35.



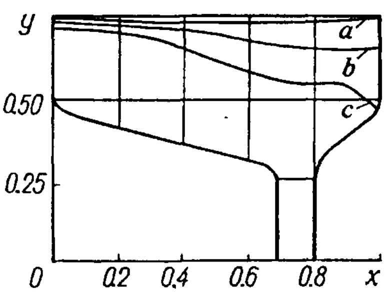


Fig. 3. Distribution of the velocity vectors in the volume of a liquid during drainage.

Fig. 4. Position of the free surface during nonsymmetric drainage from a volume: a) $t = 0.4$, b) 0.8, c) 1.2.

The remainder of the mass at the moment when the free surface reaches the drainage orifice is an important characteristic of the volume from which the liquid drains. Profiling the bottom of the volume makes it possible to decrease this remainder. Numerical calculations of drainage of a liquid from a volume with a contoured shape of the walls of the drainage orifice (Figs. 2, 3) were carried out on plane models. Results are presented for three moments of time; the dimensionless combination that characterizes the drainage of the liquid from the volume was selected to be $W = 1000$. The velocity of the liquid in the output orifice was determined in the course of calculation. Shapes of the free surface and isolines of the Cartesian components of the velocity vector are represented in Fig. 2. The isolines are drawn via the values of Δu and Δv, starting from zero values of the velocity components, which correspond to the walls of the volume. Velocity vectors are represented in Fig. 3. As the liquid moves out of the volume, a flow with a velocity profile close to parabolic forms in the output orifice. The flow that arose in the drainage orifice carries along masses of liquid located above the drainage orifice. Liquid from the periphery does not manage to fill the region near the plane symmetry; this leads to deterioration of evenness and collapse of the free surface, formation of a funnel, its touching of the drainage orifice, and breakthrough of gas into the drainage line through the drainage orifice.

The physical picture in side drainage of a liquid from a volume is characterized by great complexity. Positions of the free surface for three moments of time are presented in Fig. 4. Deviation of it toward the drainage orifice is noticeable.

CONCLUSIONS

For numerical simulation of drainage of a liquid from volumes of arbitrary cross section the Navier–Stokes equations for an incompressible liquid are used.

1. Formation of and an algorithm for solution of the initial-boundary-value problem and results of calculations of drainage from symmetric and nonsymmetric volumes have been considered.

2. Pictures of the flow of a liquid at different moments of time have been obtained. This makes it possible to evaluate the remainder of the liquid not taken in, which makes a great impact on the functioning of many technical devices of metallurgy, chemical technology, transport, etc.

NOTATION

$\hat{A}$, $\hat{B}$, Jacobi matrices of the vectors of the convective flows; $\hat{E}$, $\hat{F}$, vectors of the convective flows; J, Jacobian of the transformation of coordinates; $\hat{M}$, $\hat{N}$, Jacobi matrices of the dissipative terms; n, number of the time layer; $\hat{q}$, vector of the dependent variables; Re, Reynolds number; t, time; Δt, step in time; $\hat{T}$, $\hat{S}$, vectors of the dissipative terms; U, characteristic velocity; H, source term, depth of the volume; R, radius of the volume; r, radius of the drainage pipe; g, free-fall acceleration; u, v, components of the velocity vector in Cartesian coordinates; x, y, Cartesian coordinates; ξ, η, curvilinear coordinates linked to the surface of the volume; ξ_x, ξ_y, η_x, η_y, metrical coefficients of the transformation of coordinates; ρ, density; μ, dynamic viscosity coefficient; δ, central difference operator; Δ, ∇, operators of the left and right differences; p_0, pressure on the free surface; $\overline{V}$, velocity vector; p, pressure; β, parameter of the artificial-compressibility method; $\overline{\overline{\Pi}} = -p\overline{\overline{I}} + 2\mu\overline{\overline{E}}$, strain tensor; $\overline{\overline{I}}$, unit tensor; $\overline{\overline{E}}$, tensor of deformation rates; $\overline{n}$, $\overline{s}$, unit vectors perpendicular and tangent to the free surface; $y = f(x, t)$, equation of the free surface.

REFERENCES

1. G. V. Logvinovich, V. N. Buivol, A. S. Dudko, et al., Flows with Free Surfaces [in Russian], Kiev (1985).
2. I. B. Bogoryad, Dynamics of a Viscous Liquid with a Free Surface [in Russian], Tomsk (1980).
3. L. G. Loitsyanskii, Mechanics of Liquid and Gas [in Russian], Moscow (1987).
4. V. P. Ishchenko, G. R. Shrager, and V. A. Yakutenok, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 4, 20-25 (1988).
5. C. R. Easton and I. Catton, AIAA J., 8, No. 12, 2195-2199 (1970).
6. A. A. Prikhod'ko, Vychisl. Tekhnol., 2, No. 6, 83-93 (1993).
7. N. N. Vladimirova, B. G. Kuznetsov, and N. N. Yanenko, Certain Questions in Computational and Applied Mathematics [in a Russian], Novosibirsk, pp. 186-192 (1966).
8. P. D. Thomas and D. B. Middlecoff, Raketn. Tekh. Kosmonavtika, 18, No. 7, 55-62 (1980).
9. R. M. Beam and R. F. Warming, Raketn. Tekh. Kosmonavtika, 16, No. 4, 145-156 (1978).
10. A. A. Prikhod'ko, Modelir. Mekhanike, 3 (20), No. 6, 78-90 (1989).
11. N. M. Belyaev, O. B. Polevoi, and A. A. Prikhod'ko, Fluid Mechanics and Theory of Elasticity [in Russian], Dnepropetrovsk, pp. 16-22 (1990).
12. A. A. Prikhod'ko, Mathematical Methods of Heat and Mass Transfer [in Russian], Dnepropetrovsk, pp. 121-127 (1987).
13. A. A. Prikhod'ko, Numerical Simulation of Hydro-Gas-Dynamic Flows [in Russian], Dnepropetrovsk, pp. 95-99 (1987).